ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

CaLIGOLA Event June 09, 2023

A Quantum Day in Bologna

Cartan geometry (& Physics), A brief overview

Jordan François

Masaryk University
Faculty of Science

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Plan:

- Succinct historic background (in mathematics & physics)
- Basic modern definition (and notable features)
- Extended Cartan geometries (mapping the landscape)

Some historical background

XIXth

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1) Klein geometry (Erlangen program 1872):

Study of homogeneous a space M via its
Lie group of rigid motion G and stabiliser H.

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Study of spaces $\mathscr{M}$ locally ~ $\mathbb{E}^n$, but not globally (Riemann, Christoffel, Ricci, Levi-Civita…).

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$$\begin{array}{ccc} \text{Euclid / Minkowski} & \longrightarrow & \text{Klein geometry} \\ \downarrow & & \downarrow \\ \text{(pseudo) Riemannian} & \longrightarrow & \text{Cartan geometry} \end{array}$$

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Canadian J of Math. Vol 8, N°2 p 145-156.

ON CONNECTIONS OF CARTAN

SHÔSHICHI KOBAYASHI

Introduction. Consider a differentiable manifold M and the tangent bundle $T(M)$ over M, the structure group of which is usually the general linear group G'. Let P' be the principal fibre bundle associated with $T(M)$. Consider the fibre F of $T(M)$ as an affine space, then we have acting on F the affine transformation group G, which contains G' as the isotropic subgroup. Following the idea of Klein, it is more natural to take G as the structure group of the bundle $T(M)$. Let P be the principal fibre bundle associated to $T(M)$ with group G.

In the classical theory of affine connections, there are two points of view. The one is due to Levi-Civita, who considered each tangent space of M as a vector space and explained a connection as a law of parallel displacement of vectors along curves. From the point of view of the theory of connections in fibre bundles, a connection in the sense of Levi-Civita is a connection in the principal fibre bundle P' with group G'. The other point of view is due to E. Cartan. Following him, each tangent space of M is an affine space on which the affine transformation group G acts transitively, and an affine connection is a law of development of tangent spaces along curves; it is a connection in P.

The idea of Cartan was rigorously established by Ehresmann **(3)** as follows. Consider a fibre bundle B satisfying the conditions of *soudure* (see §2); the fibre F is homeomorphic to a homogeneous space G/G' and the structure group G of B can be reduced to G'. As in the case of tangent bundle, we obtain two principal fibre bundles P and P' with group G and G' respectively and P' is contained in P. A connection in P is called a connection of Cartan, if it satisfies the following condition: the differential form ω defining the connection gives an absolute parallelism on P'. The importance of this condition was shown in previous papers **(4; 5)**.

It is known that there is a correspondence between affine connections in the sense of Cartan and those in the sense of Levi-Civita; there is a canonical one-to-one correspondence between the set of connections in P and the set of connections in P' **(7)**.

The purpose of the present paper is to show that there exists a one-to-one correspondence between the set of Cartan connections in P and the set of infinitesimal connections in P', if the homogeneous space $F = G/G'$ is *weakly reductive* (see §2). We shall show also that in such a case the torsion forms can be defined. The last section will be devoted to the application to invariant connections.

Received April 25, 1955.

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ON CONNECTI

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145

Theory of Connections (*)

Memoria di SHOSHICHI KOBAYASCHI (a Princeton, New Jersey, U. S. A.)

Summary. - *Is given in the introduction.*

INTRODUCTION

In 1854 B. RIEMANN introduced the notion of the so-called Riemannian space, where the square of the distance of two infinitesimally near points is given by a quadratic differential form (Ueber die Hypothesen welche zu Grunde liegen).

With the notion of parallel displacement discovered by LEVI-CIVITA (1917), the first step to a new development of differential geometry was started. On the other hand, some years after the memoir of RIEMANN, F. KLEIN showed that the notion of group is at the base of geometries (Erlangen Program 1872).

For the unification of the idea of KLEIN and the notion of parallel displacement, we had to wait for the genius of E. CARTAN. He pointed

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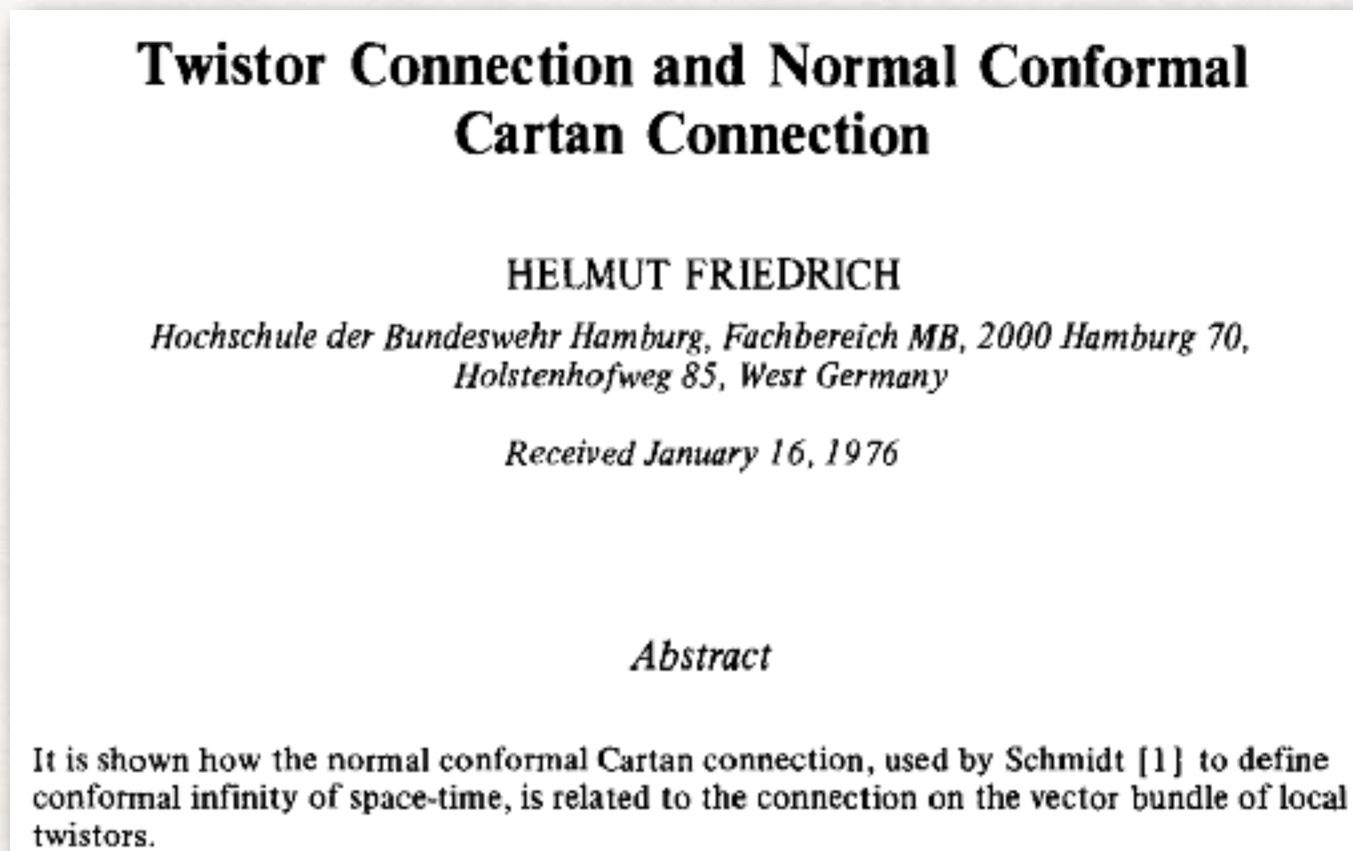

Twistor Connection and Normal Conformal Cartan Connection

HELMUT FRIEDRICH

Hochschule der Bundeswehr Hamburg, Fachbereich MB, 2000 Hamburg 70, Holstenhofweg 85, West Germany

Received January 16, 1976

Abstract

It is shown how the normal conformal Cartan connection, used by Schmidt [1] to define conformal infinity of space-time, is related to the connection on the vector bundle of local twistors.

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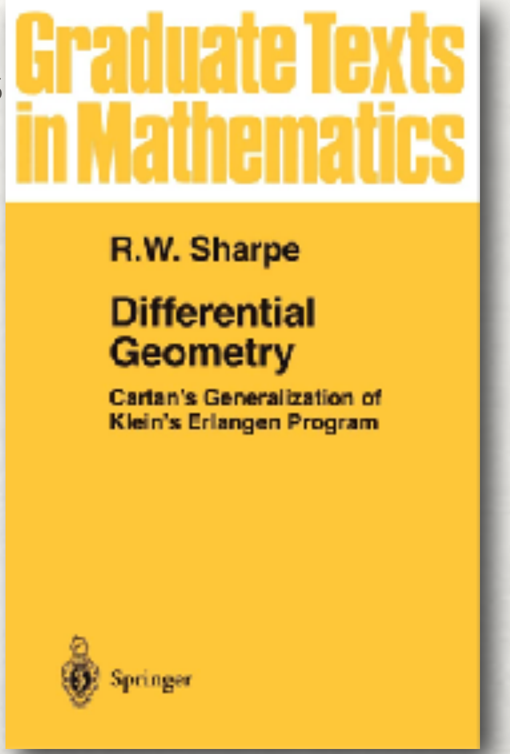

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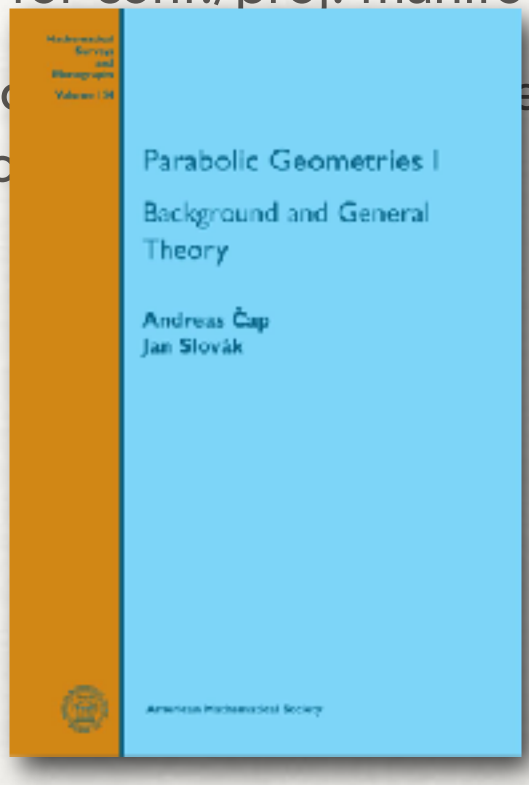

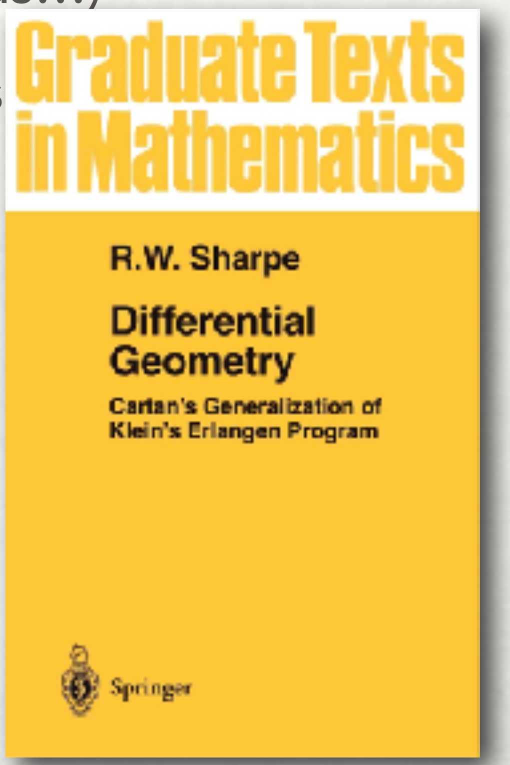

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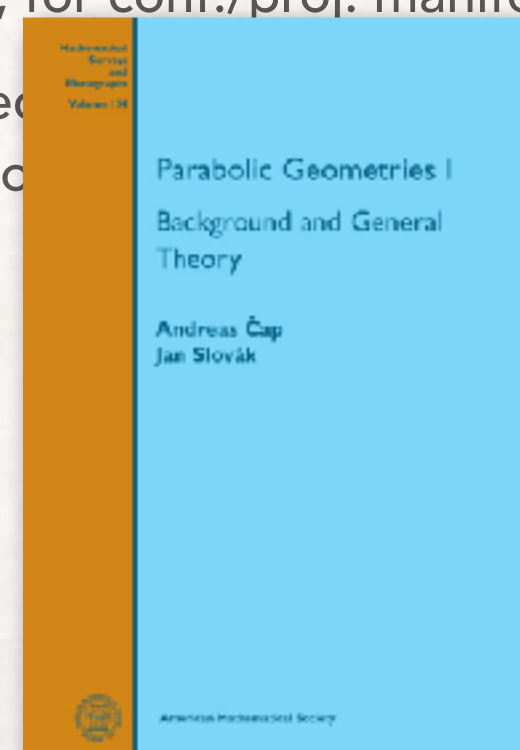
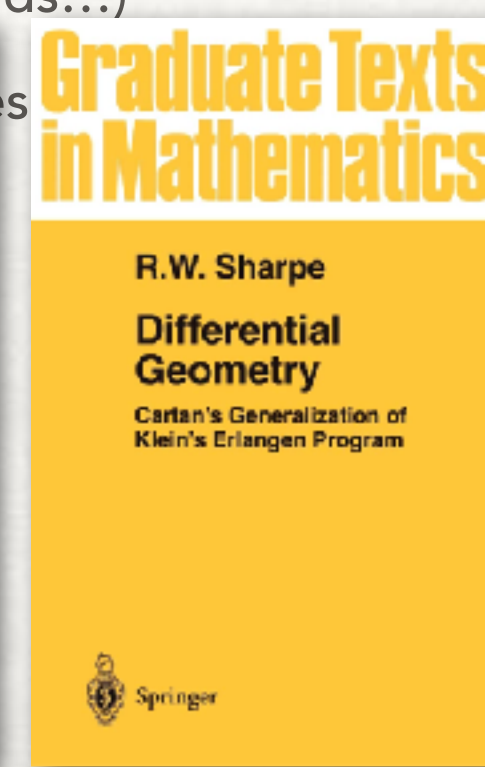
⟶ Impact on **physics**!*





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- 1922: Bach's conformal theory of gravity (Lagrangian = Weyl tensor squared).

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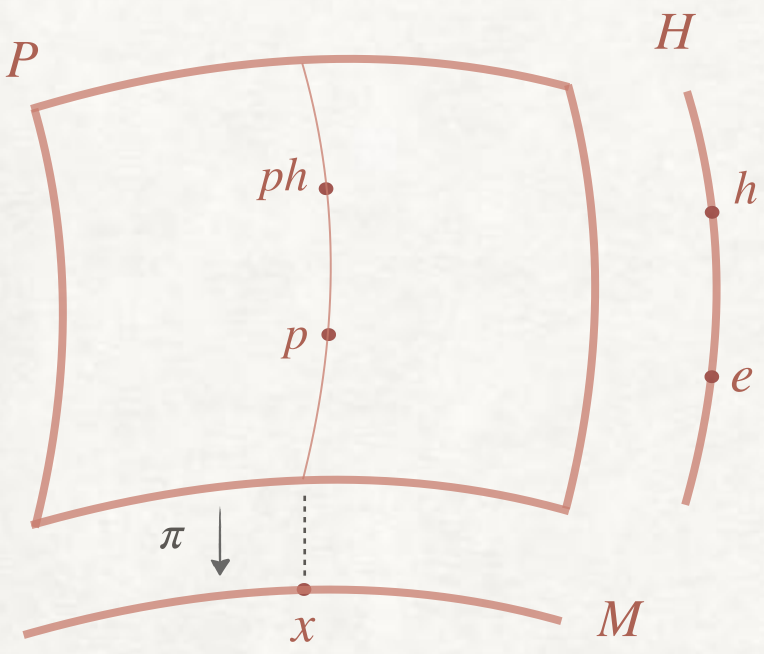
- 2000s: **Cartan geometry** as the geometric underpinning of **gauge theories of gravity.***
  Increasingly recognised by the physics community (ex: Baez, Wise 06'-09').

Principal bundle geometry



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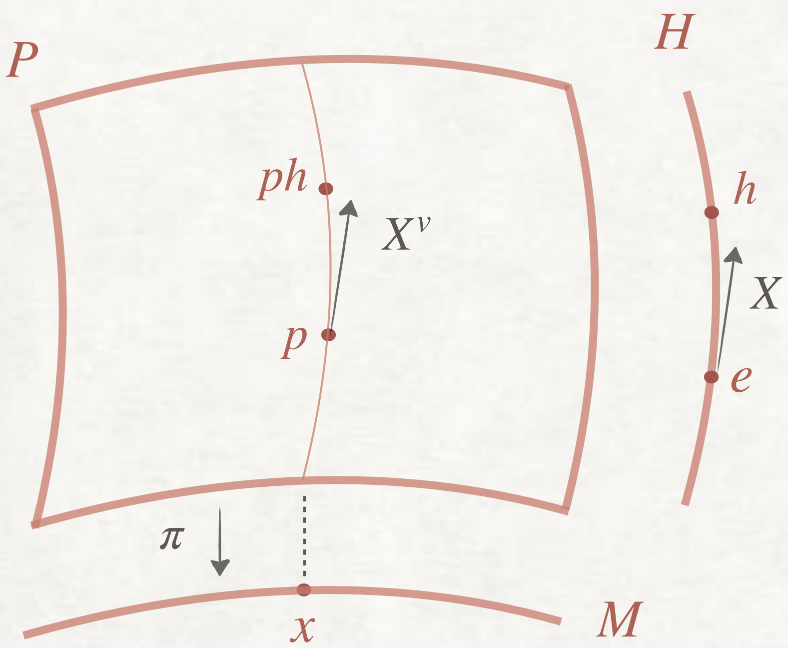

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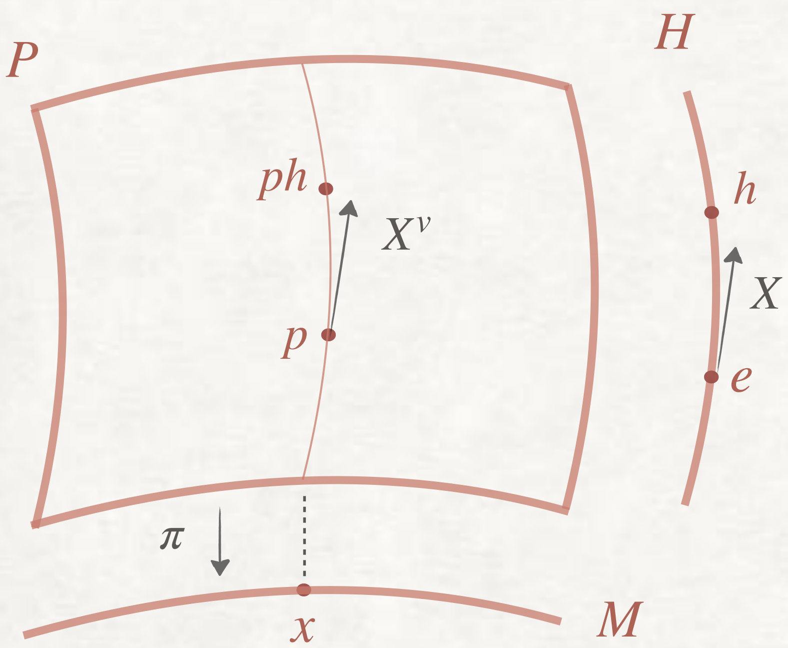

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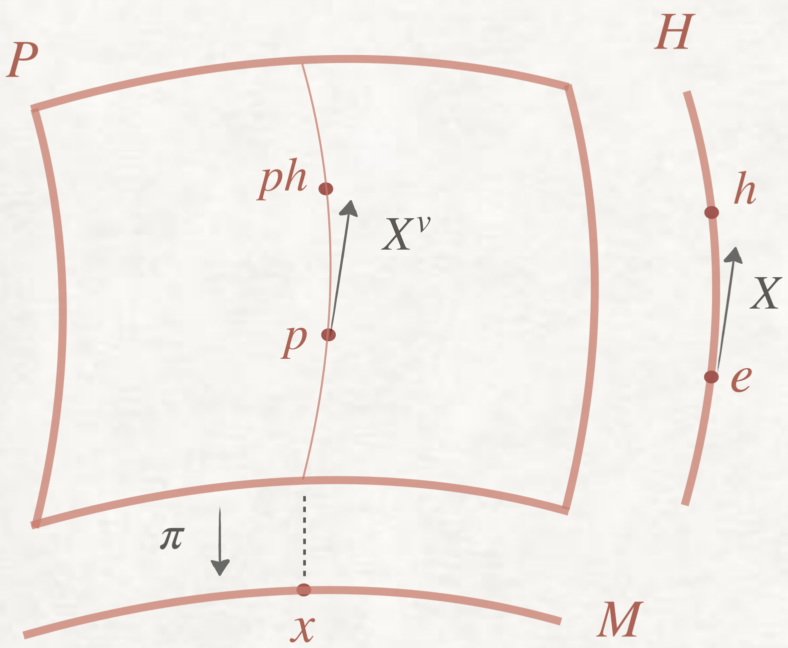

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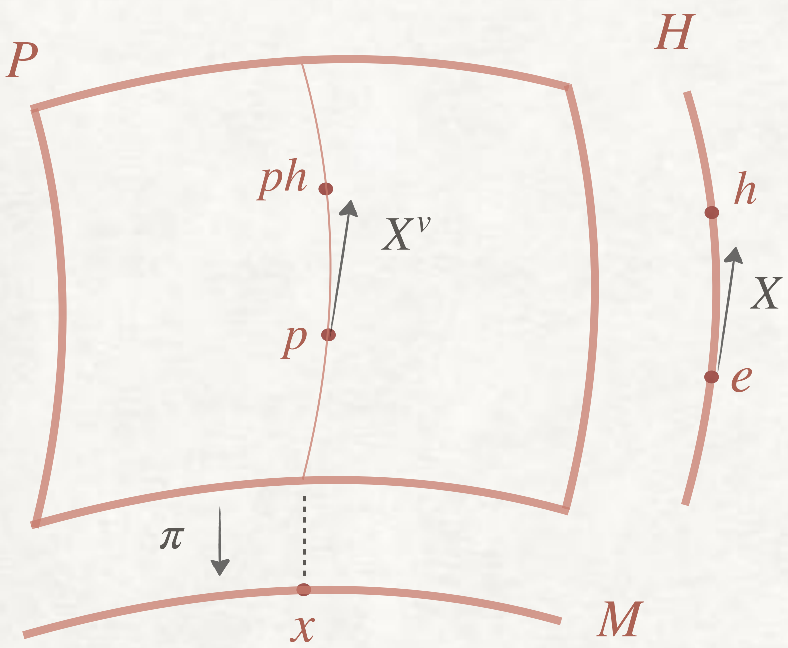

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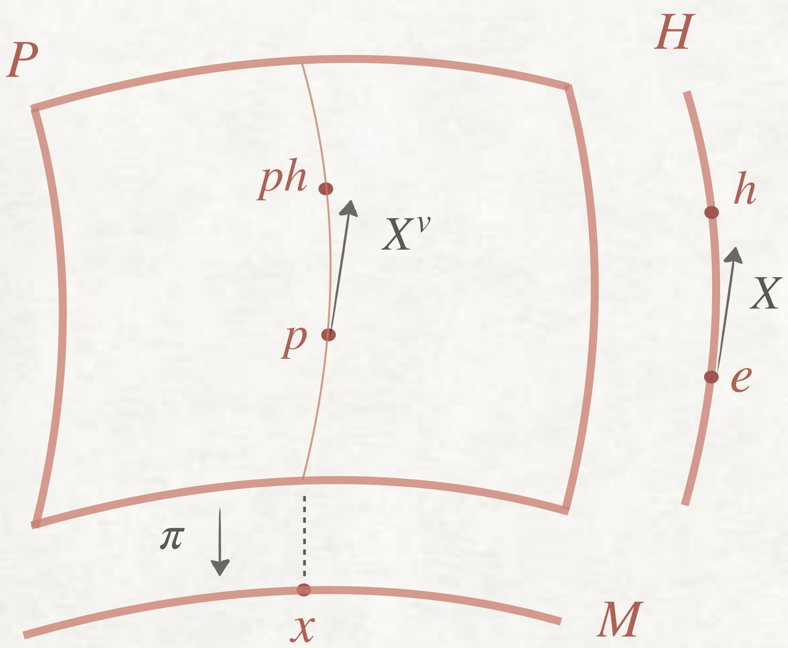

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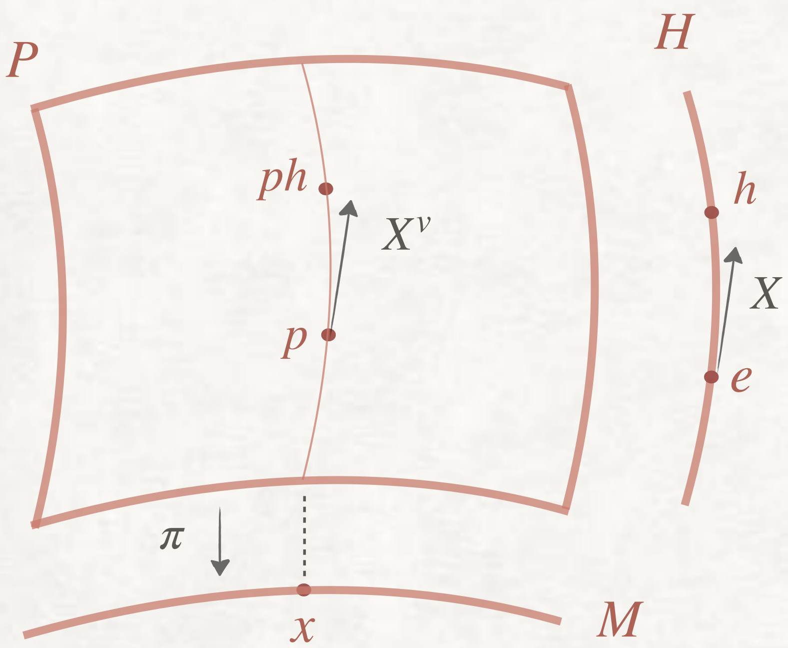

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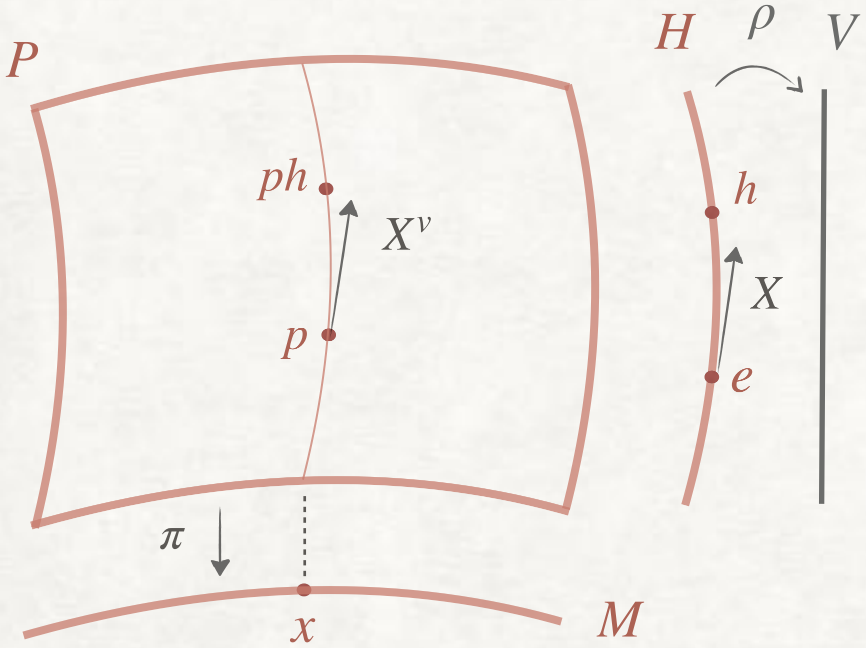

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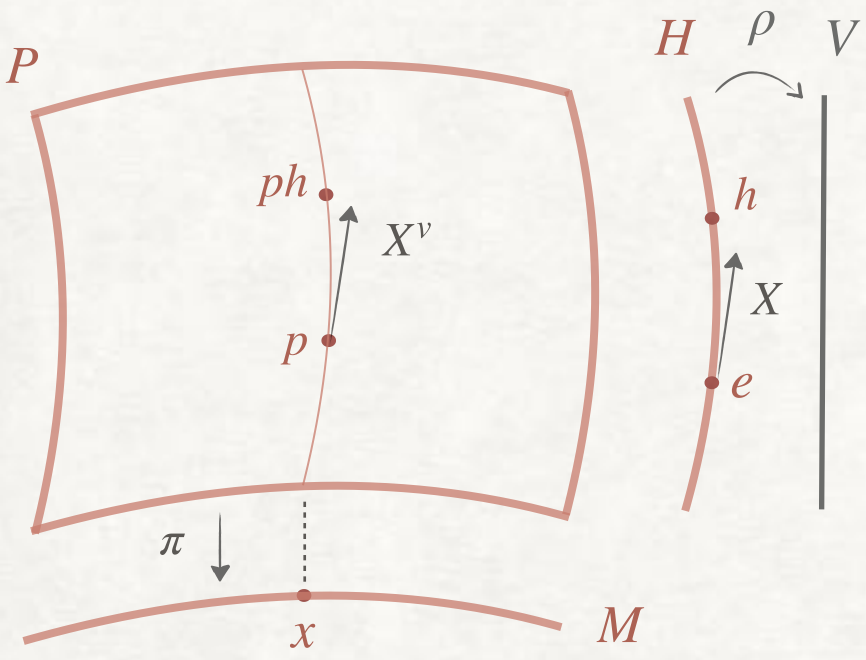

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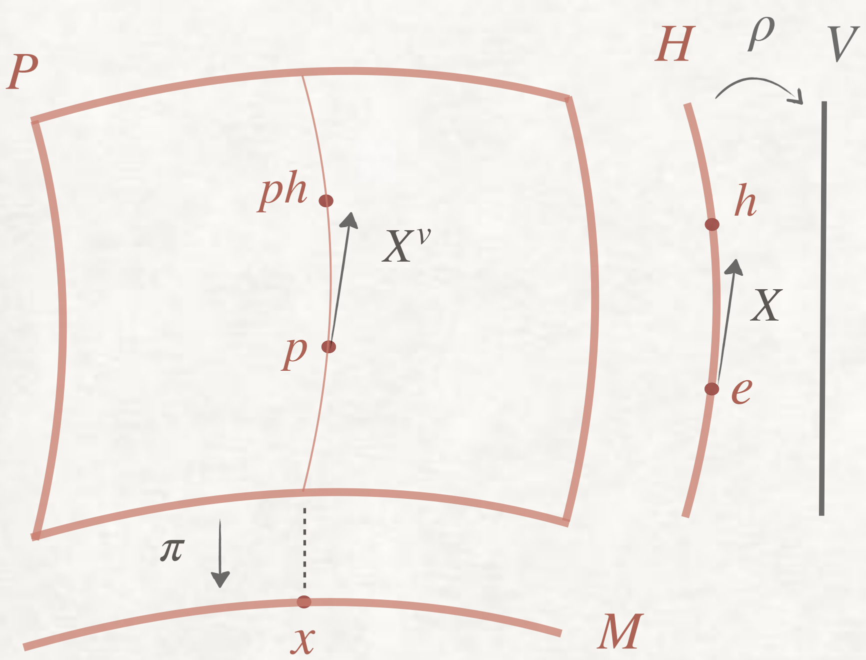

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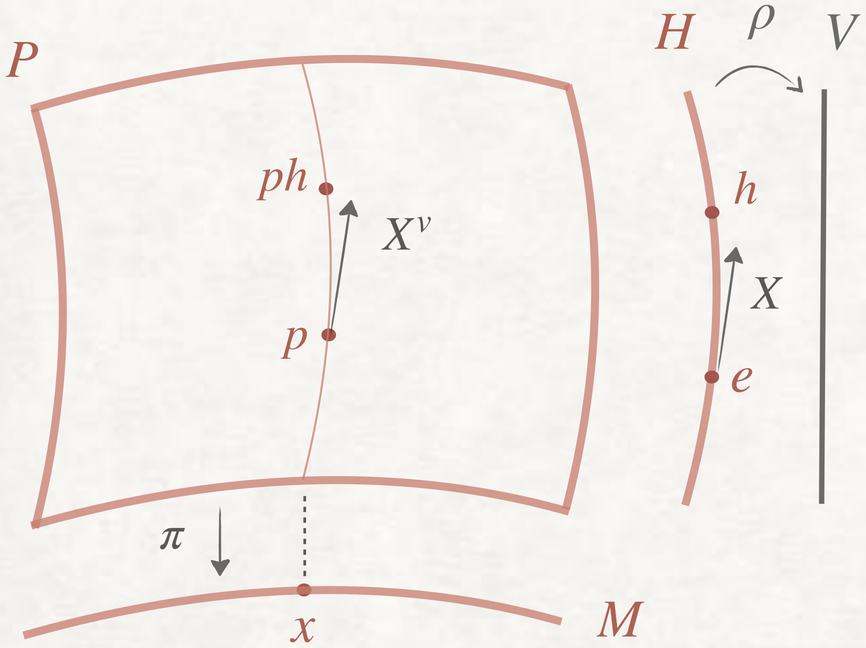

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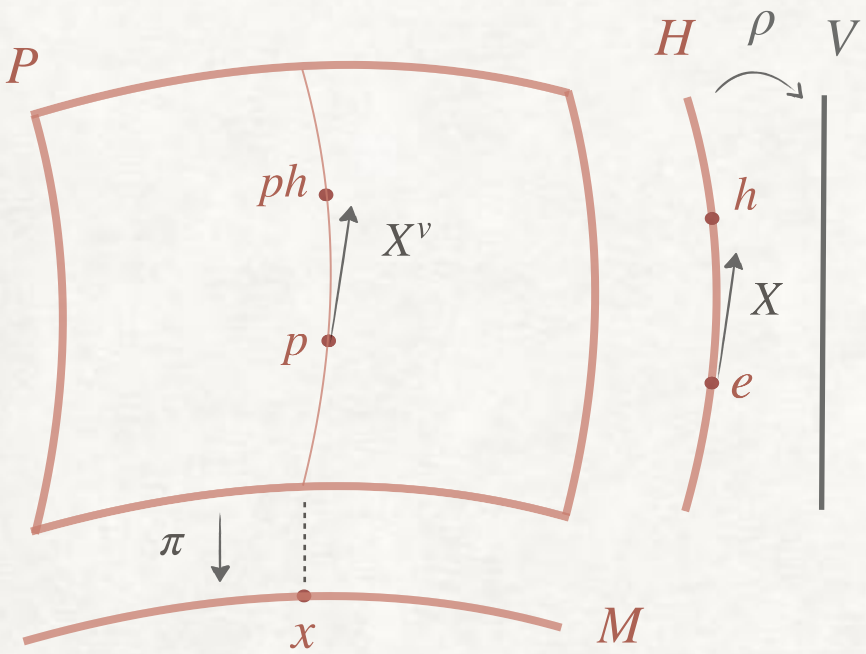

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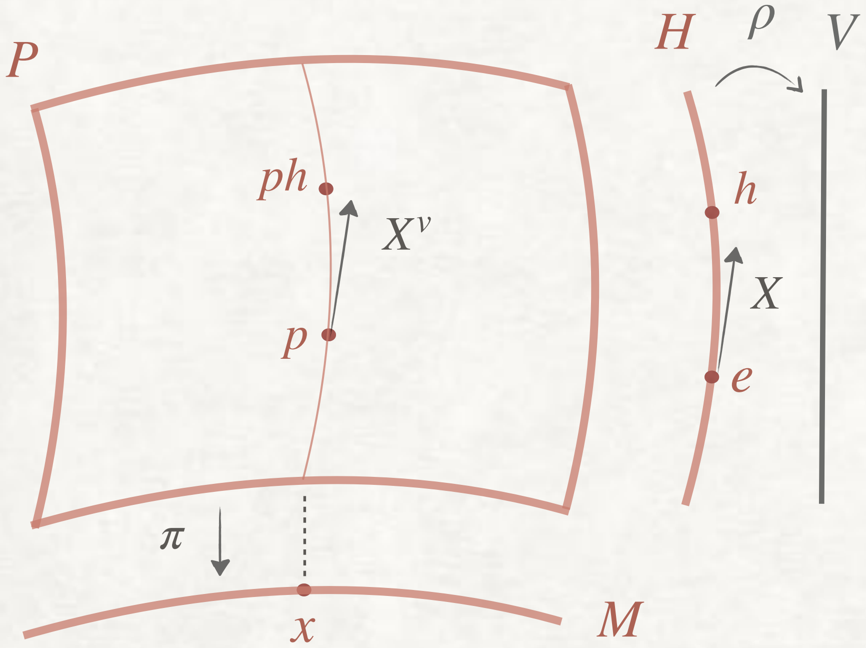

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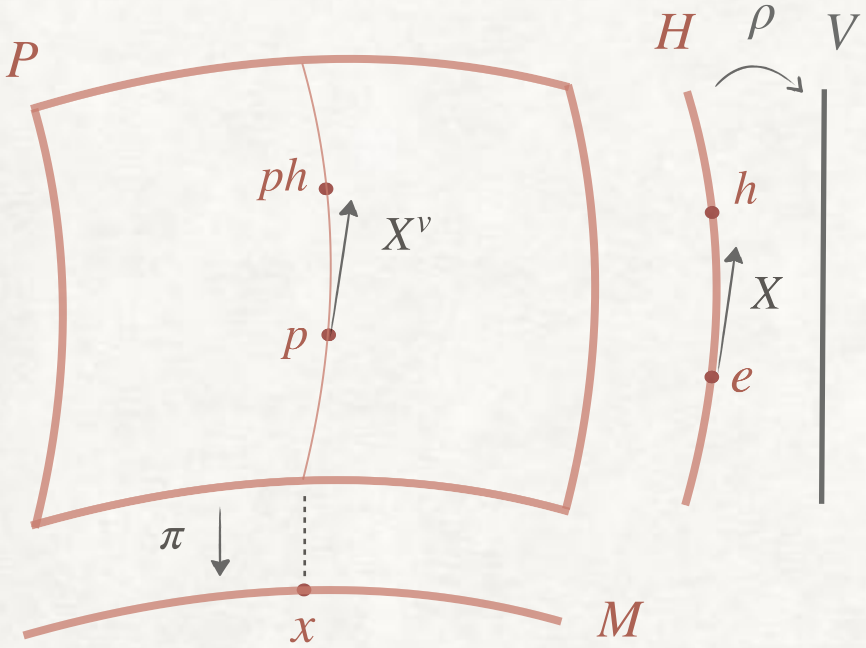

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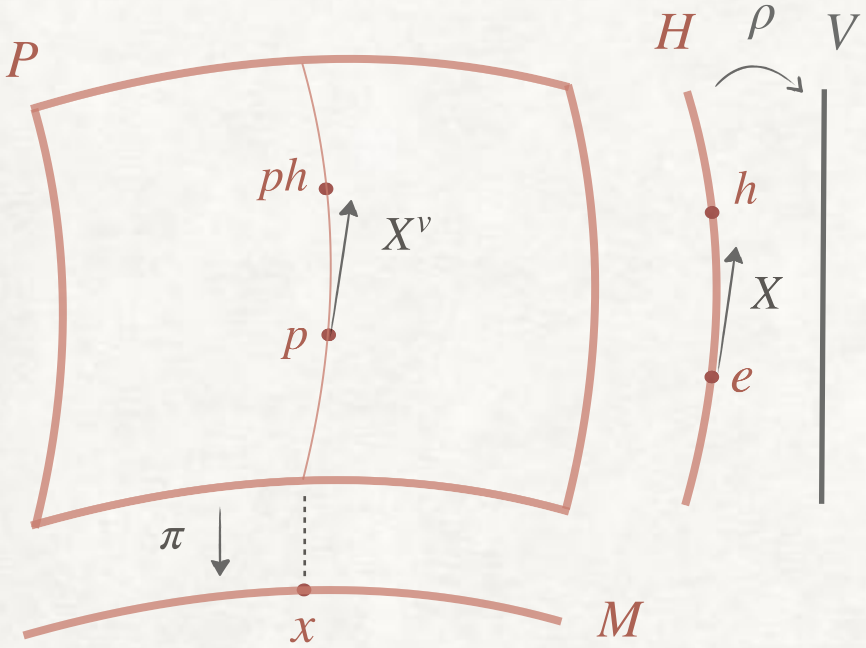

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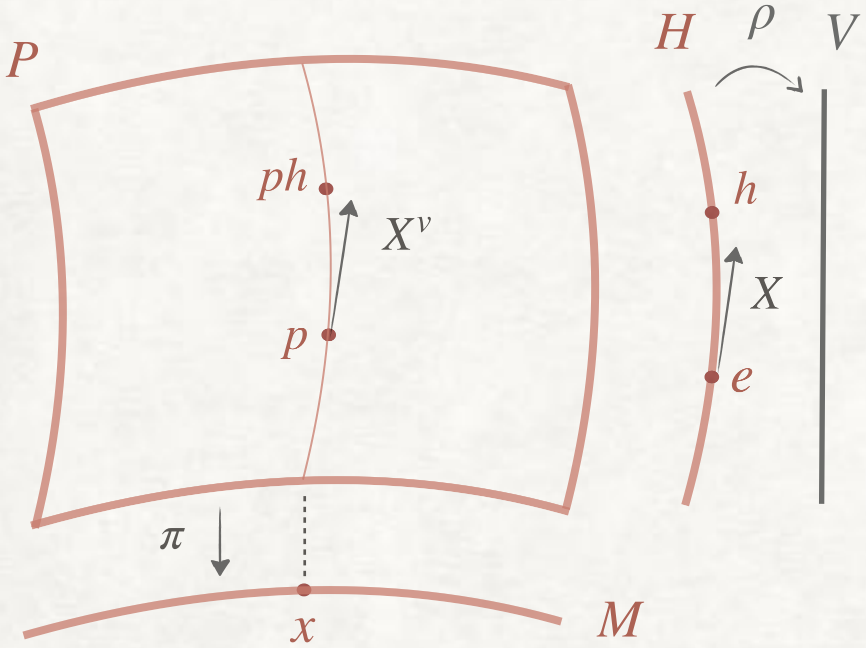

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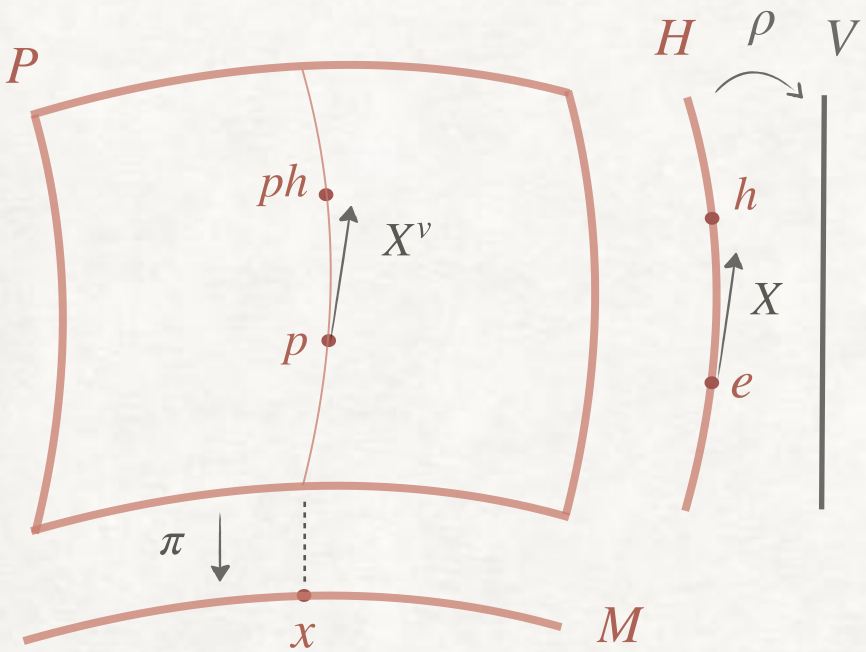
Principal bundle geometry



Transformation groups:

Bundle automorphisms $\mathrm{Aut}(P) := \{\psi \in \mathrm{Diff}(P) \mid \psi(ph) = \psi(p)h\}$

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Prob: DeRham derivative d does not preserve the space of tensorial forms. $\Longrightarrow$ Need a **connection.**

Ehresmann connection: $\omega \in \Omega^1_{eq}(P, LieH)$
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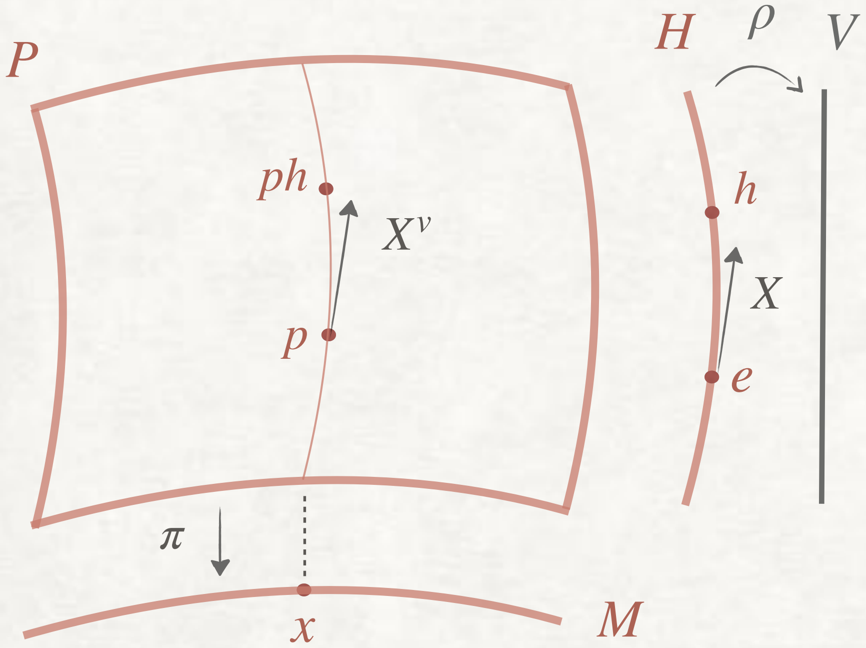

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Klein geometry: Pair of groups (G, H) s.t. G/H homogeneous space. $\Leftrightarrow$ $G \overset{H}{\rightarrow} G/H$ is a bundle.

The Maurer-Cartan form is $\varpi_G \in \Omega^1_{eq}(G, LieG)$

- $\varpi_G(X^v) = X \in LieH$
- $R_h^* \varpi_G = Ad_{h^{-1}} \varpi_G$
- $\varpi_{G|g} : T_g G \rightarrow LieG$ linear iso.

$\rightarrow$ $M \sim G/H$ homogeneous

Cartan flat $\simeq$ Klein
$\supset$ Riemann flat.

The *Maurer-Cartan equation* is: $d\varpi_G + \frac{1}{2}[\varpi_G, \varpi_G] = 0$.

$\Rightarrow$ A Klein geometry $(G \rightarrow G/H, \varpi_G)$ is a **flat** Cartan geometry! (P, ϖ) is said *modeled* on (G, ϖ_G).

Reductive geometries: $\exists$ a H-invariant splitting $LieG = LieH + V^n$ $\Rightarrow$ $\varpi = \omega + \theta$ & $\bar{\Omega} = \tilde{\Omega} + T$.

ω is a *Ehresmann connection* with curvature $\Omega \subseteq \tilde{\Omega}$ and torsion T.
The normal Cartan connection is def. by $T = 0$, so $\varpi_N = \omega(\theta) + \theta$.

Ex: The Cartan geometry (P, ϖ) with $G = H \ltimes \mathbb{R}^n$ and $H = SO(r, s)$ is (pseudo) Riemannian geometry.
(P, ϖ) with $G = SO(1, n)$ and $H = SO(1, n-1)$ is deSitter geometry.

Euclid / Minkowski	$\longrightarrow$	Klein geometry
$\downarrow$		$\downarrow$
(pseudo) Riemannian	$\longrightarrow$	Cartan geometry

Parabolic geometries: $LieG = \mathfrak{g} = \bigoplus_{-\ell \leqslant i \leqslant \ell} \mathfrak{g}_i$ s.t. $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$, and $LieH = \bigoplus_{0 \leqslant i \leqslant \ell} \mathfrak{g}_i$ (graded Lie alg.)

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Ex: Conformal $G = SO(2, n)$ & projective $G = SL(n+1)$ geometries are $\ell = 1$ - parabolic.

Extensions of Cartan geometry

Feedback physics/mathematics:

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- 70s: **Supersymmetric** gauge field theories
  = **Supergravity** gauge theories.
- 90s: Castellani-D'Auria-Fré's **superspace**
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  (Locally ringed spaces, sheaf theory)
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- À la Connes (spectral triples)
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Ex: κ-Minkowski space (q-group κ-Poincaré) ←

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NC Klein geometry = **flat** NC Cartan geometry

Curious for more… ? Come for the

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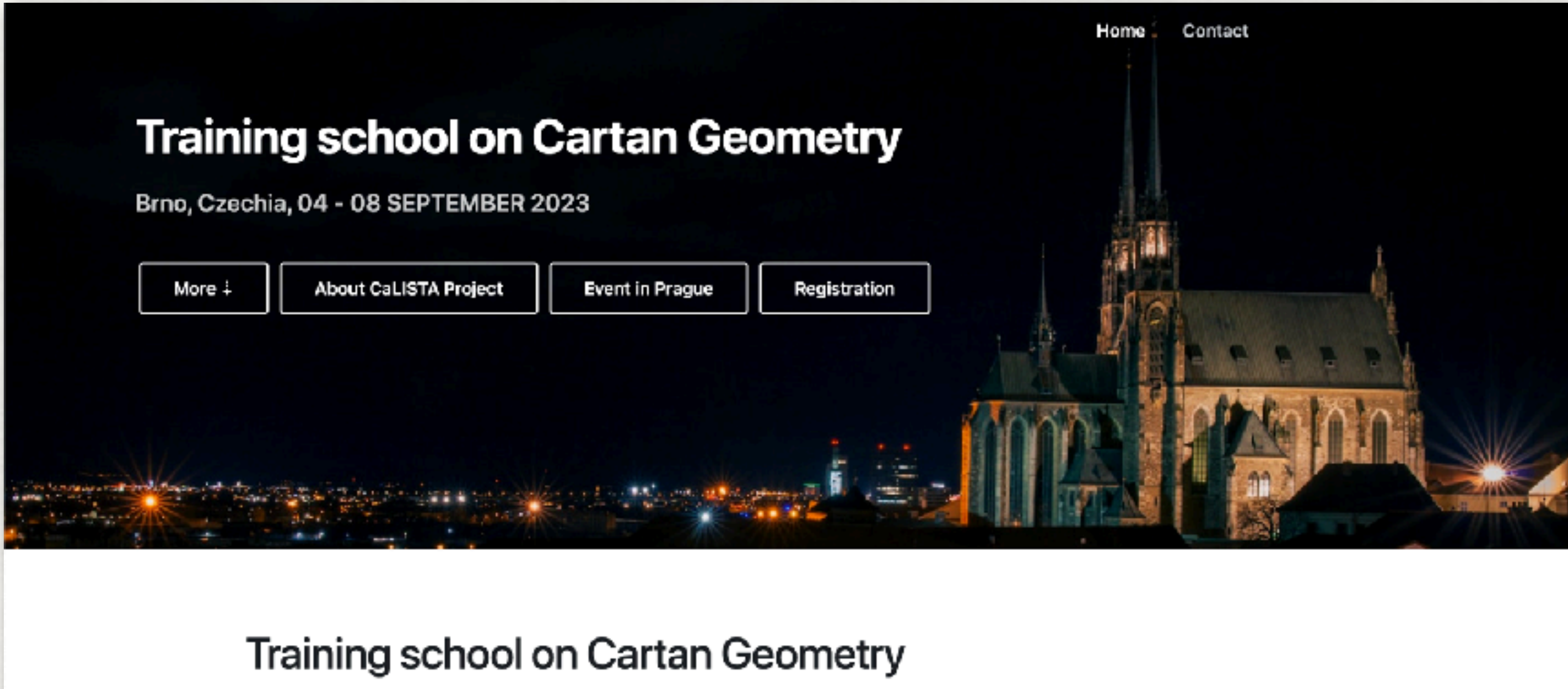

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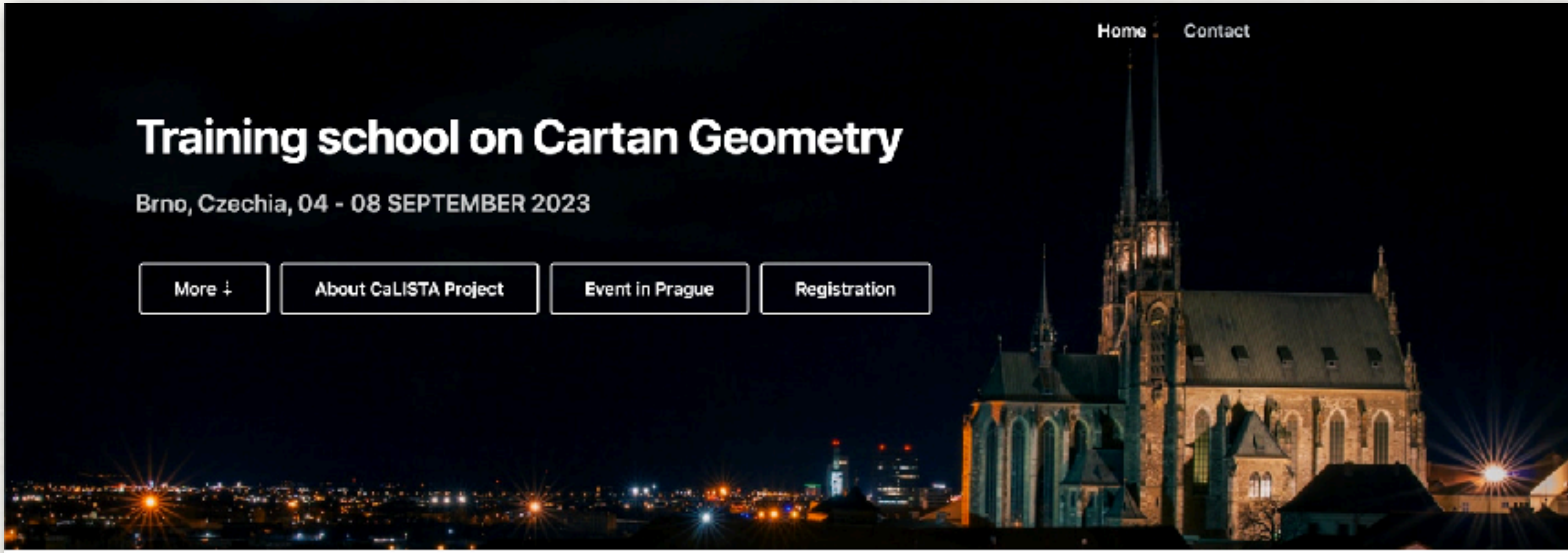

Training school on Cartan Geometry

Following speakers have agreed to provide the mini series of lectures

- **Boris Doubrov** (Minsk), **Cartan geometry via exterior calculus**
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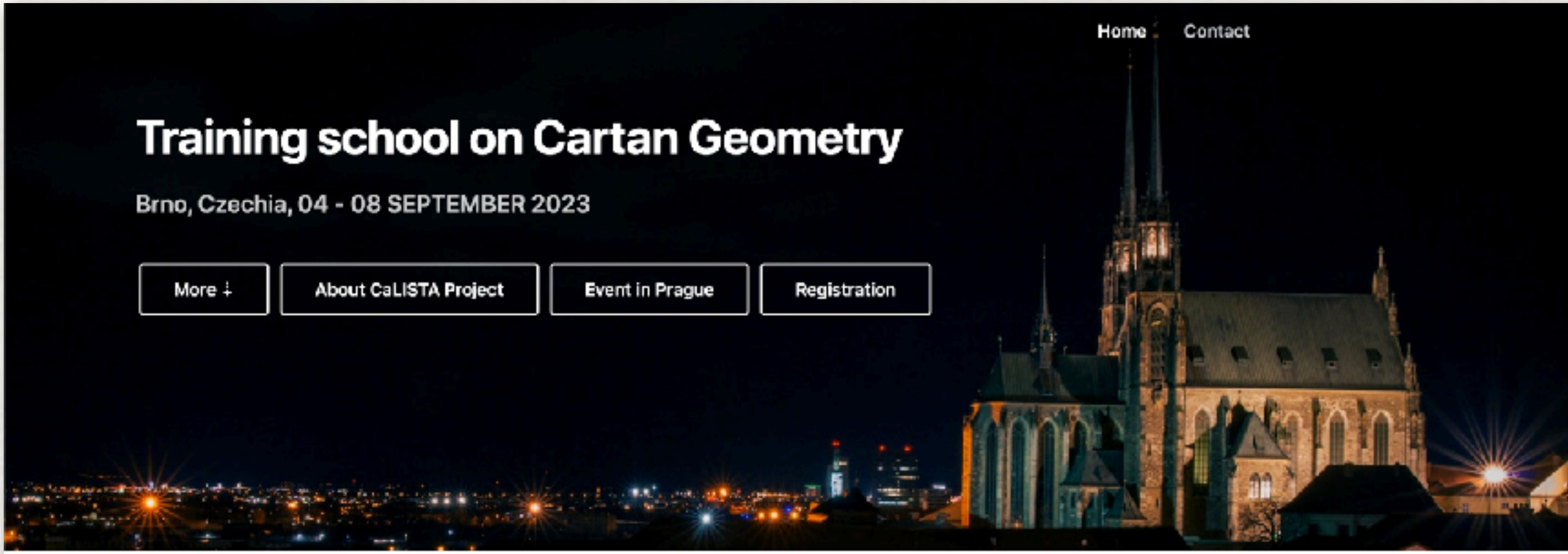

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Merci !

If ∃ a H-*invariant bilinear form* $\eta : V^n \times V^n \to \mathbb{R}$,

then $g := \eta(\theta, \theta) : \Gamma(TP) \times \Gamma(TP) \to \mathbb{R}$ *induces a metric on* M.